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The L^p-Saturation of the Bernstein–Kantorovitch Polynomials*

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In this note, we extend the saturation result of Volker Maier [2] for the Bernstein–Kantorovitch polynomials on $L^1[0, 1]$ to $L^p[0, 1]$, $1 < p < \infty$. The Bernstein–Kantorovitch polynomials are defined for $f \in L^p[0, 1]$ by

$$P_n(f, x) = \sum_{k=0}^{n} p_{nk}(x)(n+1) \int_{I_k} f(t)\, dt$$

where

$$p_{nk}(x) = \binom{n}{k} x^k (1-x)^{n-k}, \qquad I_k = \left[\frac{k}{n+1}, \frac{k+1}{n+1}\right].$$

The best direct approximation theorem for these polynomials in terms of the modulus of smoothness in $L^p[0, 1]$ has been given by Berens and DeVore [1]. However, their theorem is not invertible. We shall prove the following direct estimate.

THEOREM 1. *Let $f \in L^p[0, 1]$, $1 \leqslant p \leqslant \infty$, and suppose that f has the representation:*

$$f(x) = k + \int_{\xi}^{x} \frac{h(u)}{U}\, du \tag{*}$$

where $\xi \in (0, 1)$, $U = u(1-u)$, k is a constant, $h(0) = h(1) = 0$, and $h' \in L^p[0, 1]$, $1 < p \leqslant \infty$, or $h \in$ B.V.$[0, 1]$ for $p = 1$. Then

$$\begin{aligned}(n+1) \| P_n f - f \|_p &\leqslant C\{\| f' \|_p + \|(Xf')'\|_p\}, && 1 < p \leqslant \infty,\\ &\leqslant C\{\| Xf' \|_\infty + \| Xf' \|_{\mathrm{B.V.}}\}, && p = 1,\end{aligned}$$

where $C > 0$ is a constant and $X = x(1-x)$.

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For $p = 1$, this was proved by Maier [2], but we shall give a different derivation based on his fundamental estimates for the action of P_n on certain log functions (see Lemma 1). Maier also essentially provided (with only slight modification) the converse to Theorem 1 for $1 \leqslant p < \infty$. Thus, we have the saturation result,

THEOREM 2. (a) *If $f \in L^p[0, 1]$, $1 \leqslant p < \infty$, then f has the representation* (*) *if and only if* $(n + 1) \| P_n f - f \|_p = O(1)$.

(b) $(n + 1) \| P_n f - f \|_p = o(1)$ *if and only if f is constant a.e.*

Part (b) of Theorem 2 is a trivial consequence of the $p = 1$ result.

1. Some Lemmas

Before proving Theorem 1, we establish some estimates for three particular functions; namely, $\ln t$, $\ln(1 - t)$ and $g(t) = \ln t - \ln(1 - t)$.

LEMMA 1. *Let* $1/q = (p - 1)/p$, $1 \leqslant p < \infty$, or $1/q = 1$ for $p = \infty$.

(a) $\| x^{1/q}[P_n(\ln \cdot, x) - \ln x]\|_p = O((n + 1)^{-1})$

(b) $\| (1 - x)^{1/q}[P_n(\ln(1 - \cdot), x) - \ln(1 - x)]\|_p = O((n + 1)^{-1})$

(c) $\| X^{1/q}[P_n(g, x) - g(x)]\|_p = O((n + 1)^{-1})$.

Proof. Note that (b) follows from (a) by a change of variable and symmetry in $P_n(f, x)$. Also, (c) is a consequence of (a) and (b) and the triangle inequality.

Thus, we need only establish (a). However, Maier accomplished the essential estimates when he established (a) for $p = 1$. We shall observe that relation (a) holds for $p = \infty$. The general result then follows as in the proof of the Riesz–Thorin theorem (See Zygmund [3, p. 95].)

Volker Maier obtained the estimate

$$| P_n(\ln \cdot, x) - \ln x | \leqslant (1 - x)^n + \frac{1}{n} + \frac{5}{6} \sum_{k=1}^{n-1} \frac{p_{nk}(x)}{k^2} + \sum_{k=n+1}^{\infty} \frac{(1 - x)^k}{k} \tag{1.1}$$

([2, *p.* 48], where we have used $| r_{nn} | \leqslant n^{-1}$). Observe that $x(1 - x)^k$ has its maximum at $x = 1/(k + 1)$. Thus, the first and last terms on the right in (1.1) are $O((n + 1)^{-1})$ when multiplied by x. Similarly, $x^{k+1}(1 - x)^{n-k}$ attains its maximum at $x = (k + 1)/(n + 1)$. Therefore, applying Stirling's formula,

$$x p_{nk}(x) = O([(k + 1)/(n + 1)(n - k)]^{1/2}).$$

A careful estimate then shows that

$$\sum_{k=1}^{n-1} [(k+1)^3(n-k)(n+1)]^{-1/2} = O((n+1)^{-1}),$$

which provides the lemma.

The representation (*) implicitly contains more information about f. In fact,

LEMMA 2. *If $f \in L^p[0, 1]$, $1 < p \leqslant \infty$, and f has the representation* (*), *then $f' \in L^p[0, 1]$.*

Proof. Since $h(0) = h(1) = 0$ and $h' \in L^p[0, 1]$, we have

$$f'(x) = \frac{h(x)}{X} = \frac{1}{x}\int_0^x h'(u)\,du - \frac{1}{1-x}\int_x^1 h'(u)\,du. \tag{1.2}$$

The operators on the right in (1.2) are bounded on $L^p[0, 1]$, $1 < p \leqslant \infty$ (Hardy's inequalities).

2. PROOF OF THEOREM 1

For any $x, t \in (0, 1)$ and for f having the representation (*), there holds

$$f(t) - f(x) = Xf'(x)(g(t) - g(x)) + \int_t^x (g(u) - g(t))\,d(Uf'(u)). \tag{2.1}$$

Applying the operator P_n to (2.1) in the variable t, we obtain

$$P_n(f, x) - f(x) = Xf'(x)[P_n(g, x) - g(x)] + P_n\left(\int_t^x (g(u) - g(t))\,d(Uf'(u)), x\right). \tag{2.2}$$

Taking $L^p[0, 1]$ norms on both sides and applying Lemmas 1(c) and 2, we obtain

$$\| P_n(f, x) - f(x)\|_p \leqslant \frac{C}{n+1}\|f'\|_p + \left\| P_n\left(\int_t^x (g(u) - g(t))\,d(Uf')(u)), x\right)\right\|_p \tag{2.3}$$

(for $p = 1$, $\|f'\|_p$ is replaced by $\|Xf'\|_\infty$). Thus, it remains to bound the second term on the right in (2.3). We do this for $p = 1$ and $p = \infty$ and then use interpolation theory.

LEMMA 3. $(n+1)\| P_n(\int_t^x (g(u) - g(t))\,dh(u), x)\|_1 \leqslant c\| h\|_{\mathrm{B.V.}}$

Proof. Let $K(n, t, x) = \sum_{k=0}^{n} p_{nk}(x)(n+1)\chi_{I_k}(t)$. Then using the fact that $\operatorname{sgn}(g(u) - g(t)) = \operatorname{sgn}(u - t)$ and using Fubini's theorem twice, we obtain

$$\begin{aligned}
\int_0^1 \left| P_n \left(\int_t^x (g(u) - g(t))\, dh(u), x \right) \right| dx
&\leqslant \int_0^1 \int_0^x K(n, t, x) \int_t^x (g(u) - g(t))\, |dh(u)|\, dt\, dx \\
&\quad + \int_0^1 \int_x^1 K(n, t, x) \int_x^t (g(t) - g(u)))\, |dh(u)|\, dt\, dx \\
&\leqslant \int_0^1 \left\{ \int_u^1 \int_0^u K(n, t, x)(g(u) - g(t))\, dt\, dx \right. \\
&\quad \left. + \int_0^u \int_u^1 K(n, t, x)(g(t) - g(u))\, dt\, dx \right\} |dh(u)| \\
&= \int_0^1 \left\{ \int_u^1 P_n((g(u) - g(\cdot))_+, x)\, dx \right. \\
&\quad \left. + \int_0^u P_n((g(\cdot) - g(u))_+, x)\, dx \right\} |dh(u)|.
\end{aligned}$$

Hence, we must show that

$$\int_u^1 P_n((g(u) - g(\cdot))_+, x)\, dx + \int_0^u P_n((g(\cdot) - g(u))_+, x)\, dx = O((n+1)^{-1}). \tag{2.4}$$

But, $(g(u) - g(x))_+ = 0$ on $u \leqslant x \leqslant 1$, and for any function f, $\int_0^1 (P_n(f, x) - f(x))\, dx = 0$. Therefore,

$$\begin{aligned}
\int_u^1 P_n((g(u) - g(\cdot))_+, x)\, dx
&= -\int_0^u [P_n((g(u) - g(\cdot))_+, x) - (g(u) - g(x))_+]\, dx.
\end{aligned}$$

Hence, the left-hand side of (2.4) equals

$$\begin{aligned}
\int_0^u [P_n((g(\cdot) - g(u)), x) - (g(x) - g(u))]\, dx
&= \int_0^u [P_n(g, x) - g(x)]\, dx \\
&= O((n+1)^{-1})
\end{aligned}$$

by Lemma 1(c) for $p = 1$.

LEMMA 4. $(n+1)\| P_n(\int_t^x (g(u)-g(t))\, h'(u)\, du, x)\|_\infty \leqslant C \| h' \|_\infty$.

Proof. As for the last lemma, we have

$$\left| P_n \left(\int_t^x (g(u)-g(t))\, h'(u)\, du, x \right) \right|$$

$$= \left| \int_0^x K(n,t,x) \int_t^x (g(u)-g(t))\, h'(u)\, du\, dt \right.$$

$$\left. + \int_x^1 K(n,t,x) \int_x^t (g(t)-g(u))\, h'(u)\, du\, dt \right|$$

$$\leqslant \| h' \|_\infty \int_0^1 K(n,t,x) \int_t^x (g(u)-g(t))\, du\, dt$$

$$= \| h' \|_\infty \int_0^1 K(n,t,x) \{[\ln x - \ln t] x$$

$$+ (1-x)[\ln(1-x) - \ln(1-t)]\}\, dt$$

$$= \| h' \|_\infty \{x\{\ln x - P_n(\ln \cdot, x)] + (1-x)[\ln(1-x) - P_n(\ln(1-\cdot), x)]\}$$

$$= \| h' \|_\infty\, O((n+1)^{-1})$$

by Lemma 1(a) and (b).

In order to complete the proof of Theorem 1, we observe that Lemmas 3 and 4 imply that the linear operator

$$T_n F(x) = (n+1)\, P_n \left(\int_t^x (g(u)-g(t))\, F(u)\, du, x \right)$$

is bounded independently of n on $L^p[0,1]$ for $p = 1, \infty$. Thus, it is bounded for all p, $1 \leqslant p \leqslant \infty$. Taking $F(u) = (Uf'(u))'$, we see that the theorem follows.

Remark. By Lemma 2, $f' \in C(0,1) \cap L^p[0,1]$, which implies that $|f'(x)| = O(X^{-1/p})$. Thus, in (2.3) and the statement of Theorem 1, we could replace $\| f' \|_p$ by $\| X^{1/p} f' \|_\infty$ when $1 < p \leqslant \infty$.

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